PROCEEDINGS

Workshop on Integrable Field Theories, Solitons and Duality

Recursion operators of the N=2 supersymmetric unconstrained matrix GNLS hierarchies

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Abstract: A super–algebraic formulation of the $N = 2$ supersymmetric unconstrained matrix $(k|n, m)$–MGNLS hierarchies (nlin.SI/0201026) is established. Recursion operators, fermionic and bosonic symmetries as well as their superalgebra are constructed for these hierarchies.

1. Introduction

The $N = 2$ supersymmetric unconstrained matrix $(k|n, m)$–Generalized Nonlinear Schödinger ($(k|n, m)$–MGNLS) hierarchies were proposed in [1] by exhibiting the corresponding *matrix pseudo–differential* Lax–pair representation in terms of $N = 2$ *unconstrained superfields* for the bosonic isospectral flows. These hierarchies generalize and contain as limiting cases many other interesting $N = 2$ supersymmetric hierarchies discussed in the literature. When matrix entries are chiral and antichiral $N = 2$ superfields, these hierarchies reproduce the $N = 2$ chiral matrix $(k|n, m)$-GNLS hierarchies [2, 3], and in turn the latter coincide with the $N = 2$ GNLS hierarchies of references [4, 5] in the scalar case $k = 1$. When matrix entries are unconstrained $N = 2$ superfields and $k = 1$, these hierarchies are equivalent to the $N = 2$ supersymmetric multicomponent hierarchies [6]. The bosonic limit of the $N = 2$ unconstrained $(k|0, m)$–MGNLS hierarchy reproduces the bosonic matrix NLS equation elaborated in [7] via the $gl(2k+m)/(gl(2k) \times gl(m))$–coset construction. The $N = 2$ $(1|1, 0)$–MGNLS hierarchy is related to one of three different existing $N = 2$

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supersymmetric KdV hierarchies – the $N = 2$ $\alpha = 1$ KdV hierarchy – by a reduction [6, 1, 8].

Apart from the Lax–pair representation for the isospectral bosonic flows of the $N = 2$ unconstrained $(k|n,m)$–MGNLS hierarchies, at present we do not know other quantities and/or data (if any) which could characterize their integrable structure, like, e.g. their super–algebraic formulation, bosonic and fermionic symmetries, Hamiltonian structures, recursion operators, etc. (although part of these are known for some of above–mentioned limiting cases).

The present talk addresses these problems. We obtain a super–algebraic formulation of the $N = 2$ unconstrained $(k|n,m)$–MGNLS hierarchies. Using it and the superalgebraic methods developed in refs. [9, 10, 11, 12, 13, 14, 15] and especially [16] we derive the superalgebra of fermionic and bosonic symmetries as well as the recursion operators for these hierarchies.

The paper is organized as follows. In Section 2.1 we present a short summary of the pseudo–differential Lax–pair approach to the $N = 2$ unconstrained $(k|n,m)$–MGNLS hierarchies. In Section 2.2 we rewrite the corresponding spectral equation in a local matrix form and establish its super–algebraic structure which is then used in Section 2.3 and 2.4 to derive the superalgebra of the symmetries and the recursion operators of the hierarchy, respectively. In Section 2.5 we discuss supersymmetry and locality of the isospectral flows. In Section 3 we summarize our results and discuss open problems.

2. The $N = 2$ unconstrained $(k|n,m)$–MGNLS hierarchies

2.1 Pseudo–differential Lax pair representation

The Lax–pair representation for the bosonic flows of the $N = 2$ supersymmetric unconstrained $(k|n,m)$–MGNLS hierarchies is [1]

$$\tfrac{\partial}{\partial t_p} L = [A_p, L], \quad L = I\partial + FD\overline{D}\partial^{-1}\overline{F}, \quad A_p = (L^p)_{\geq 0} + res(L^p), \quad p \in \mathbb{N} \tag{2.1}$$

where the subscript ≥ 0 denotes the sum of purely differential and constant parts of the operator L^p, and $res(L^p)$ is its $N = 2$ supersymmetric residue, i.e. the coefficient of $[D,\overline{D}]\partial^{-1}$. Here, $F \equiv F_{Aa}(Z)$ and $\overline{F} \equiv \overline{F}_{aA}(Z)$ $(A, B = 1,\dots,k;\ a, b = 1,\dots,n+m)$ are rectangular matrices which entries are unconstrained $N = 2$ superfields, I is the unity matrix, $I_{AB} \equiv \delta_{AB}$, and the matrix product is implied, for example $(F\overline{F})_{AB} \equiv \sum_a F_{Aa}\overline{F}_{aB}$. The matrix entries are Grassmann even superfields for $a = 1,\dots,n$ and Grassmann odd superfields for $a = n+1,\dots,n+m$. Thus, fields do not commute, but rather satisfy $F_{Aa}\overline{F}_{bB} = (-1)^{d_a\overline{d}_b}\overline{F}_{bB}F_{Aa}$ where d_a and $\overline{d}_b$ are the Grassmann parities of the matrix elements F_{Aa} and $\overline{F}_{bB}$, respectively, $d_a = 1$ $(d_a = 0)$ for odd (even) entries. Fields depend on the coordinates $Z = (z,\theta,\overline{\theta})$ of $N = 2$ superspace. The volume element in superspace is $dZ \equiv dzd\theta d\overline{\theta}$. Finally, $D, \overline{D}$ are the $N = 2$ supersymmetric fermionic covariant derivatives

$$D = \frac{\partial}{\partial\theta} - \frac{1}{2}\overline{\theta}\frac{\partial}{\partial z}, \quad \overline{D} = \frac{\partial}{\partial\overline{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial z}, \quad D^2 = \overline{D}^2 = 0, \quad \{D,\overline{D}\} = -\frac{\partial}{\partial z} \equiv -\partial. \tag{2.2}$$

The algebra of the flows in (2.1) is abelian

$$[\frac{\partial}{\partial t_m}, \frac{\partial}{\partial t_n}] = 0. \tag{2.3}$$

The Lax pair representation (2.1) may be seen as the integrability condition for the following linear system:

$$L\psi_1 = \lambda\psi_1, \tag{2.4}$$

$$\frac{\partial}{\partial t_p}\psi_1 = A_p\psi_1 \tag{2.5}$$

where λ is the spectral parameter and the eigenfunction ψ_1 is the Baker-Akhiezer function of the hierarchy.

2.2 Matrix formulation of the spectral equation

Let us rewrite the spectral equation (2.4) in a matrix form in $N = 2$ superspace

$$\mathcal{L}\Psi = 0, \quad \overline{\mathcal{L}}\Psi = 0, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{pmatrix} \tag{2.6}$$

with two $N = 2$ odd Lax operators $\mathcal{L}$ and $\overline{\mathcal{L}}$

$$\mathcal{L} = D + A_\theta, \quad \overline{\mathcal{L}} = \overline{D} + A_{\overline{\theta}} \tag{2.7}$$

whose odd connections A_θ and $A_{\overline{\theta}}$ are restricted to be local functionals of the original superfield matrices F and $\overline{F}$ and their $\{D, \overline{D}\}$–derivatives. One finds that the eigenvalue equation (2.4) is equivalent to (2.6) provided the connections are chosen as

$$A_\theta = \Lambda + A, \quad A_{\overline{\theta}} = \overline{\Lambda} + \overline{A},$$

$$\Lambda = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \overline{\Lambda} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ \lambda & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \end{pmatrix},$$

$$A = 0, \quad \overline{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -F & 0 & 0 \\ \overline{D}\,\overline{F} & 0 & 0 & \mathcal{I}\overline{F} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ F\mathcal{I}\overline{D}\,\overline{F} & 0 & -\overline{D}F & F\overline{F} & 0 \end{pmatrix} \tag{2.8}$$

where Λ and $\overline{\Lambda}$ are constant matrices and we have introduced the notation

$$\mathcal{I}_{ab} := (-1)^{\overline{d}_a}\delta_{ab}. \tag{2.9}$$

Using eqs. (2.7–2.8) one can derive the even Lax operator

$$\mathcal{L}_z := -(\widehat{\mathcal{L}}\,\overline{\mathcal{L}} + \widehat{\overline{\mathcal{L}}}\mathcal{L}) = \partial - \lambda E + \mathcal{A}, \quad \mathcal{L}_z\Psi = 0 \tag{2.10}$$

where the transformation $\mathcal{L} \to \widehat{\mathcal{L}}$ simply amounts to a change in the sign of the Grassmann-even matrix entries in $\mathcal{L}$ and

$$E = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & 0 & F & 0 & 0 \\ 0 & 0 & DF & 0 & 0 \\ -D\overline{D}\,\overline{F} & \overline{D}\mathcal{I}\overline{F} & 0 & -D\mathcal{I}\overline{F} & -\overline{F} \\ -F\overline{D}\mathcal{I}\overline{F} & 0 & \overline{D}F & -F\overline{F} & 0 \\ -DF\overline{D}\mathcal{I}\overline{F} & F\overline{D}\mathcal{I}\overline{F} & D\overline{D}F & -DF\overline{F} & -F\overline{F} \end{pmatrix}. \tag{2.11}$$

One important remark is in order: *the connection $\mathcal{A}$ (2.10–2.11) does not depend on the spectral parameter λ, and this property is crucial for the construction that will be carried out.* We would like to emphasize that there are infinitely many representations equivalent to (2.8) which generically do not possess this property[1]. The representation (2.8) is just adapted to use the approach developed in [14, 15, 16] (see also references therein).

Remark: there is, however, another representation with properties analogous to the representation just described. We consider the matrix

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ F & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \lambda & 0 & -\overline{F} & 0 & 1 \end{pmatrix}, \tag{2.12}$$

and transform the operators in (2.7) and (2.10) to

$$\mathcal{L}' = \widehat{K}\mathcal{L}K^{-1} = D + \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -D\overline{F} & -\mathcal{I}\overline{F} & 0 & 0 & 0 \\ \lambda + F\overline{F} & 0 & -F & 0 & -1 \\ -(DF)\overline{F} & -\lambda & DF & 0 & 0 \end{pmatrix},$$

$$\overline{\mathcal{L}}\,' = \widehat{K}\overline{\mathcal{L}}K^{-1} = \overline{D} + \begin{pmatrix} 0&0&0&-1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix},$$

$$\mathcal{L}'_z = K\mathcal{L}_zK^{-1} = \partial - \lambda E + \begin{pmatrix} -F\overline{F} & 0 & F & 0 & 0 \\ -(DF)\overline{F} & 0 & DF & 0 & 0 \\ \overline{D}D\overline{F} & \overline{D}\mathcal{I}\overline{F} & 0 & -D\mathcal{I}\overline{F} & \overline{F} \\ \overline{D}(F\overline{F}) & 0 & \overline{D}F & -F\overline{F} & 0 \\ \overline{D}((DF)\overline{F}) & 0 & -\overline{D}DF & -(DF)\overline{F} & 0 \end{pmatrix}. \tag{2.13}$$

[1] In other words, in most cases the spectral parameter appears in field-dependent terms.

All matrix entries in formulae (2.8), (2.11) and (2.13) are rectangular or square blocks. For instance, all 1's stand for the $k \times k$ identity matrix. A short inspection shows that after interchanging the components ψ_2 and ψ_5 ($\psi_2 \leftrightarrow \psi_5$) of Ψ (2.6), the matrices in $\mathcal{L}_z$ and $\mathcal{L}'_z$ have the following block structure

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \left(\begin{array}{c|c} \underset{(2k+n)\times(2k+n)}{even} & \underset{(2k+n)\times(2k+m)}{odd} \\ \hline \underset{(2k+m)\times(2k+n)}{odd} & \underset{(2k+m)\times(2k+m)}{even} \end{array}\right) \tag{2.14}$$

and zero supertrace

$$Str \begin{pmatrix} A & B \\ C & D \end{pmatrix} := Tr(A) - Tr(D) = 0, \tag{2.15}$$

so that they belong to the superalgebra

$$\mathcal{G} = sl(2k+n|2k+m). \tag{2.16}$$

The constant matrix E (2.11) defines the splitting

$$\begin{gathered} \mathcal{G} = Ker(ad_E) \oplus Im(ad_E), \quad E^2 = E, \\ Ker(ad_E) = \begin{pmatrix} * & * & 0 & * & * \\ * & * & 0 & * & * \\ 0 & 0 & * & 0 & 0 \\ * & * & 0 & * & * \\ * & * & 0 & * & * \end{pmatrix}, \quad Im(ad_E) = \begin{pmatrix} 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ * & * & 0 & * & * \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 \end{pmatrix} \end{gathered} \tag{2.17}$$

which possesses the properties

$$\begin{aligned} & [Ker(ad_E),\ Ker(ad_E)\} \in Ker(ad_E), \\ & [Ker(ad_E),\ Im(ad_E)\} \in Im(ad_E), \\ & [Im(ad_E),\ Im(ad_E)\} \in Ker(ad_E), \\ & (ad_E)^2\Big|_{Im(ad_E)} = I\Big|_{Im(ad_E)} \end{aligned} \tag{2.18}$$

and

$$Ker(ad_E) = s\Big(gl(2k|2k) \oplus gl(n|m)\Big). \tag{2.19}$$

In what follows we will use the homogeneous gradation of the loop superalgebra

$$\mathcal{G} \otimes C[\lambda, \lambda^{-1}] \tag{2.20}$$

with the grading operator

$$d = \lambda \tfrac{\partial}{\partial \lambda}. \tag{2.21}$$

The matrices λE and $\mathcal{A}$ (2.11) entering into the even Lax operator $\mathcal{L}_z$ (2.10) belong to the subspaces with grades 1 and 0 respectively

$$[d, \lambda E] = \lambda E, \quad [d, \mathcal{A}] = 0. \tag{2.22}$$

We shall construct a non–local gauge transformation G, which commutes with E, $GEG^{-1} = E$ and which is fixed by the requirement that it transforms $\mathcal{A}$ in (2.11) to a connection $\widetilde{\mathcal{A}}$ belonging to $Im(ad_E)$.

$$\widetilde{\mathcal{A}} = G\mathcal{A}G^{-1} + G\partial G^{-1}, \quad \widetilde{\mathcal{A}} \in Im(ad_E). \tag{2.23}$$

With this aim let us first define a $k \times k$ matrix g, which will be useful in what follows, by the consistent set of equations

$$\partial g = -gF\overline{F}, \quad Dg = -\left(\partial^{-1} g(DF\overline{F})g^{-1}\right)g, \quad \overline{D}g = -\left(\partial^{-1} g(\overline{D}F\overline{F})g^{-1}\right)g. \tag{2.24}$$

Hereafter, we also use the notation

$$\begin{aligned} &f := \partial^{-1} gF\overline{D}\mathcal{I}\overline{F}, \\ &\overline{Q} := \overline{D} - g^{-1}f, \\ &\widehat{\overline{Q}}\ \overline{F} := \overline{D}\ \overline{F} + \mathcal{I}\overline{F}g^{-1}f. \end{aligned} \tag{2.25}$$

Then, the relevant gauge transformation turns out to be

$$\Psi \quad \Rightarrow \quad \widetilde{\Psi} = G\Psi, \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -f & 0 & 0 & g & 0 \\ -Df & f & 0 & Dg & g \end{pmatrix} \tag{2.26}$$

and the corresponding even and odd matrix Lax operators become

$$\widetilde{\mathcal{L}} = G\mathcal{L}G^{-1} = D + \Lambda, \quad \widetilde{\overline{\mathcal{L}}} = G\overline{\mathcal{L}}G^{-1} = \overline{D} + \widetilde{\overline{\Lambda}} + \widetilde{\overline{\mathcal{A}}}, \quad \widetilde{\mathcal{L}}\widetilde{\Psi} = \widetilde{\overline{\mathcal{L}}}\widetilde{\Psi} = 0,$$

$$\widetilde{\overline{\Lambda}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \end{pmatrix},$$

$$\widetilde{\overline{\mathcal{A}}} = \begin{pmatrix} -g^{-1}f & 0 & 0 & -g^{-1} & 0 \\ Dg^{-1}f & -g^{-1}f & -F & Dg^{-1} & g^{-1} \\ \widehat{\overline{Q}}\ \overline{F} & 0 & 0 & \mathcal{I}\overline{F}g^{-1} & 0 \\ g\overline{Q}g^{-1}f & 0 & 0 & g\overline{Q}g^{-1} & 0 \\ -Dg\overline{Q}g^{-1}f & -g\overline{Q}g^{-1}f & -g\overline{Q}F & -Dg\overline{Q}g^{-1} & g\overline{Q}g^{-1} \end{pmatrix} \tag{2.27}$$

and

$$\widetilde{\mathcal{L}}_z = G\mathcal{L}_z G^{-1} = \partial - \lambda E + \widetilde{\mathcal{A}}, \quad \widetilde{\mathcal{L}}_z \widetilde{\Psi} = 0,$$

$$\widetilde{\mathcal{A}} = \begin{pmatrix} 0 & 0 & F & 0 & 0 \\ 0 & 0 & DF & 0 & 0 \\ -D\widehat{\overline{Q}} & \overline{F} & \widehat{\overline{Q}}\mathcal{I}\overline{F} & 0 & -D\mathcal{I}\overline{F}g^{-1} & -\overline{F}g^{-1} \\ 0 & 0 & g\overline{Q}F & 0 & 0 \\ 0 & 0 & Dg\overline{Q}F & 0 & 0 \end{pmatrix} \in Im(ad_E), \tag{2.28}$$

respectively.

2.3 Flows

Now, following ref. [16] one can define flows of the hierarchy corresponding to the matrix Lax operator (2.28)

$$D_{X_p}\widetilde{\mathcal{L}}_z = [(X_p^{\widetilde{\Theta}})_+, \widetilde{\mathcal{L}}_z], \quad X_p^{\widetilde{\Theta}} = \widetilde{\Theta}\lambda^p X \widetilde{\Theta}^{-1}, \quad X \in Ker(ad_E), \quad p \in \mathbb{N}^+ \tag{2.29}$$

where D_{X_p} denote the corresponding evolution derivatives, $\widetilde{\Theta}$ is the dressing matrix defined by

$$\widetilde{\Theta}^{-1}\left(\partial - \lambda E + \widetilde{\mathcal{A}}\right)\widetilde{\Theta} = \partial - \lambda E, \tag{2.30}$$

and the subscript + denotes the projection on the positive homogeneous grading (2.21). The algebra of the flows (2.29) is isomorphic to the superalgebra

$$\widehat{Ker}(ad_E) := Ker(ad_E) \otimes P(\lambda), \tag{2.31}$$

where $P(\lambda)$ is the set of polynomials in the spectral parameter λ. The isospectral flows $\frac{\partial}{\partial t_p}$ (2.1) of the hierarchy, forming an abelian algebra (2.3), have to be generated by the central element $X = E$ of the kernel $Ker(ad_E)$ via equations (2.29). All other flows from the set (2.29) by construction commute with the isospectral flows and form their bosonic and fermionic symmetries (for detail, see [16]). To close this subsection let us only mention that the subalgebra $sl(2k|2k) \otimes P(\lambda) \subset \widehat{Ker}(ad_E)$ of the symmetry algebra (2.31) contains two different odd symmetries which may be seen as extensions of the $N = 2$ supersymmetry algebra. Two possible choices are obtained from the matrices

$$X_{p\pm}^{(1)} = \begin{pmatrix} 0 & \lambda^p & 0 & 0 & 0 \\ \pm\lambda^{p+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^p \\ 0 & 0 & 0 & \pm\lambda^{p+1} & 0 \end{pmatrix}, \quad X_{p\pm}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & \lambda^p & 0 \\ 0 & 0 & 0 & 0 & \lambda^p \\ 0 & 0 & 0 & 0 & 0 \\ \pm\lambda^{p+1} & 0 & 0 & 0 & 0 \\ 0 & \pm\lambda^{p+1} & 0 & 0 & 0 \end{pmatrix}. \tag{2.32}$$

satisfying the anticommutation relations

$$\{X_{p\pm}^{(i)},\ X_{k\pm}^{(i)}\} = \pm 2\lambda^{p+k+1}E, \quad \{X_{p-}^{(i)},\ X_{k+}^{(i)}\} = 0, \quad i = 1, 2 \tag{2.33}$$

The existence of a similar rich symmetry structure for the particular case of the reduced $N = 2$ unconstrained $(1|1,0)$–MGNLS hierarchy was observed recently in [8].

2.4 Recursion operators

Using the general formula for recurrence relations

$$\frac{\partial}{\partial t_p}\widetilde{\mathcal{A}} = \Big(\partial - ad_{\widetilde{\mathcal{A}}}\, \partial^{-1}\, ad_{\widetilde{\mathcal{A}}}\Big)\, ad_E\, \frac{\partial}{\partial t_{p-1}}\widetilde{\mathcal{A}} \tag{2.34}$$

derived in [14, 16] for the case of Hermitian symmetric spaces (2.18), we obtain the following recurrence relations for the hierarchy under consideration with the Lax operator $\widetilde{\mathcal{L}}_z$ (2.28):

$$\begin{aligned}
\frac{\partial}{\partial t_p}F \;=&\; \frac{\partial}{\partial t_{p-1}}F\,' + F\partial^{-1}\frac{\partial}{\partial t_{p-1}}D\overline{D}\;\overline{F}F - Dg^{-1}\Big(\frac{\partial}{\partial t_{p-1}}g\Big)\overline{Q}F + D\Big(\partial^{-1}\frac{\partial}{\partial t_{p-1}}F\widehat{\overline{Q}}\mathcal{I}\overline{F}\Big)F \\
&- \Big(\partial^{-1}\frac{\partial}{\partial t_{p-1}}(DF)\widehat{\overline{Q}}\mathcal{I}\overline{F}\Big)F - \Big(\partial^{-1}\frac{\partial}{\partial t_{p-1}}(DF)\overline{F}g^{-1}\Big)g\overline{Q}F, \\
\frac{\partial}{\partial t_p}(\overline{F}g^{-1}) \;=&\; -\frac{\partial}{\partial t_{p-1}}(\overline{F}g^{-1})\,' - \Big(\partial^{-1}\frac{\partial}{\partial t_{p-1}}D\overline{D}\;\overline{F}F\Big)\overline{F}g^{-1} - (D\widehat{\overline{Q}}\;\overline{F})\Big(\frac{\partial}{\partial t_{p-1}}g^{-1}\Big) \\
&+ (\widehat{\overline{Q}}\mathcal{I}\overline{F})\Big(\partial^{-1}\frac{\partial}{\partial t_{p-1}}(DF)\overline{F}g^{-1}\Big) - (D\mathcal{I}\overline{F}g^{-1})\Big(\partial^{-1}\frac{\partial}{\partial t_{p-1}}g(\overline{Q}F)\overline{F}g^{-1}\Big) \\
&- \overline{F}g^{-1}\Big(\partial^{-1}\frac{\partial}{\partial t_{p-1}}(Dg\overline{Q}F)\overline{F}g^{-1}\Big)
\end{aligned} \tag{2.35}$$

where $'$ denotes the derivative with respect to the space variable z.

We have verified explicitly by direct calculations that the first few bosonic flows generated by eqs. (2.35) with the initial recursion step

$$\frac{\partial}{\partial t_1}F = F\,', \quad \frac{\partial}{\partial t_1}\overline{F} = \overline{F}\,' \tag{2.36}$$

reproduce the corresponding isospectral flows $\frac{\partial}{\partial t_p}$ of the $N=2$ supersymmetric unconstrained $(k|n,m)$–MGNLS hierarchy resulting from the pseudo–differential Lax–pair representation (2.1).

2.5 Supersymmetry and locality

Although at the component level, the non-zero matrix entries in the connection $\widetilde{\mathcal{A}}$ in (2.28) are all independent, this is not so at the superfield level. The connection satisfies constraints, which may be written as

$$[\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}_z\} = \widetilde{\mathcal{L}}\widetilde{\mathcal{L}}_z - \widehat{\widetilde{\mathcal{L}}}_z\widetilde{\mathcal{L}} = 0, \quad [\widetilde{\overline{\mathcal{L}}}, \widetilde{\mathcal{L}}_z\} = \widetilde{\overline{\mathcal{L}}}\widetilde{\mathcal{L}}_z - \widehat{\widetilde{\mathcal{L}}}_z\widetilde{\overline{\mathcal{L}}} = 0. \tag{2.37}$$

If these constraints are respected by the flows, then the flows are consistent with supersymmetry. In fact, only the first of these constraints is easily shown to be respected by the isospectral flows. Using the dressing equation (2.30), we rewrite this constraint as

$$[\widehat{\widetilde{\Theta}}^{-1}(D+\Lambda)\widetilde{\Theta}, \partial - \lambda E\} = 0. \tag{2.38}$$

Considering this equation at each homogeneous gradation, it is easy to show that it leads to

$$\widehat{\widetilde{\Theta}}^{-1}(D+\Lambda)\widetilde{\Theta} = D + \Lambda. \tag{2.39}$$

It is then clear that the matrix $E_p^{\widetilde{\Theta}} = \widetilde{\Theta}\lambda^p E\widetilde{\Theta}^{-1}$ commutes with the operator $D+\Lambda$. Since this last operator respects the homogeneous gradation, we end up with the equation

$$[D+\Lambda, (E_p^{\widetilde{\Theta}})_+\} = 0, \tag{2.40}$$

which shows that the isospectral flows respect the first of constraints (2.37). We conjecture that the second of these constraints is also preserved, although we could not show it.

Let us discuss shortly the locality of the isospectral flows (2.29) with $X = E$. When rewritten in terms of the local operator $\mathcal{L}_z$ in (2.10), they become

$$D_{E_p}\mathcal{L}_z = [(E_p^{\Theta})_+ - G^{-1}D_{E_p}G, \mathcal{L}_z], \quad E_p^{\Theta} = \Theta\lambda^p E\Theta^{-1}, \quad p \in \mathbb{N}^+, \tag{2.41}$$

where the matrix Θ is obtained from dressing the operator $\mathcal{L}_z$

$$\Theta^{-1}\Big(\partial - \lambda E + \mathcal{A}\Big)\Theta = \partial - \lambda E. \tag{2.42}$$

It is known that the matrix $(E_p^{\Theta})_+$ is a local functional in the fields and their derivatives. Moreover, from the form of G in (2.26) one can show that the second term $-G^{-1}D_{E_p}G$ of the Lax representation (2.41) does not contribute to the field equation of F, which is thus local. This is not so, however, for $\overline{F}$.

We conjecture that in order to demonstrate completely the supersymmetry and locality of the isospectral flows, one should make use of the second representation introduced in (2.13). This point is still under investigation.

3. Conclusion

In this paper we have constructed a $sl(2k+n|2k+m)$–super–algebraic formulation (2.27–2.28) of the $N=2$ supersymmetric unconstrained $(k|n,m)$–MGNLS hierarchies in $N=2$ superspace. Then we have derived the superalgebra $s\Big(gl(2k|2k)\oplus gl(n|m)\Big)\otimes P(\lambda)$ (2.31) of their fermionic and bosonic symmetries (2.29). We have observed that this symmetry superalgebra contains many odd flows, some of them generalizing the $N=2$ supersymmetry algebra. Finally we have constructed the recursion relations (2.35) for these hierarchies.

Let us finish this paper with a few questions for the future. It is easily seen that the connection $\widetilde{\mathcal{A}}$ entering into the Lax operator $\widetilde{\mathcal{L}}_z$ (2.28) is nonlocal. Moreover, its $N=2$ superfield entries are not independent[2] quantities, i.e. they are subjected to constraints. Why in this case do isospectral matrix flows (2.29) be local, as it is obviously the case in their original pseudo–differential representation (2.1)? Why are they supersymmetric, or in other words, why do these flows preserve the above–mentioned constraints? Finally, how can one see in general that these flows coincide with the original flows (2.1) we started with. These questions are clarified only partly in the present paper, and we hope to discuss them in more detail elsewhere.

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[2]Though, at the component level, the entries in $\widetilde{\mathcal{A}}$ (2.28) are independent, and fill up the image $Im(ad_E)$.

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